In completing the assignment, you ***cannot*** *use Large Language Models like ChatGPT or a calculator.*

The assignment is designed to build conceptual understanding and guide you towards the right problem-solving approach. Getting the right answer is less important than the questions you ask. There are **no** solutions. Try your best and send it to me for feedback. There is **no** due date.

Problem Set 1

Problem 1. *Can you write an algebraic expression that denotes the set of all points* (x, y) *belonging to the boundary of a circle centred at* (x_0, y_0) *of radius* r*? Can you explain what the expression means in words?*

Solution: The boundary of the circle is the set of all points (x, y) satisfying the equation

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

In words, this simply says that the points (x, y) are a fixed distance r from the point (x_0, y_0). This is just Pythagoras' Theorem (try drawing a picture to justify this to yourself).

Actually, you can take this one step further to say that the points belonging to the disc (i.e. containing the boundary as well as the points inside the circle) can be described by the inequality

$$(x - x_0)^2 + (y - y_0)^2 \leq r^2.$$

Problem 2. *What are the coordinates of the apex of the parabola composed of all points* (x, y) *satisfying the expression*

$$y = ax^2 + bx + c.$$

Give an expression in terms of a, b *and* c.

Solution: I've listed a couple of ways you can get to the solution. Note that I'm trying to keep a visual model in mind to help make sense of the algebra. Sometimes, keeping a visual picture of what's going on—and not just blindly computing the algebra—will help you see the forest through the trees.

Method 1. We know (using our visual imagination) that the apex occurs when the parabola is at its highest (if it's concave) or lowest (if it's convex). So we need to work out (algebraically) when the parabola is at its extreme points in the y-axis.

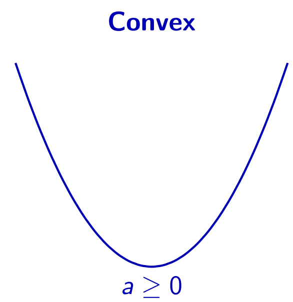
If we complete the square we can rewrite the equation as

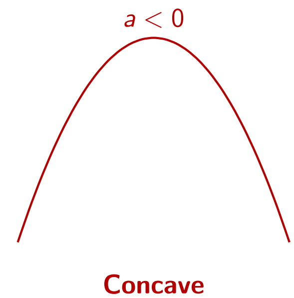
$$\begin{aligned} \frac{y}{a} &= x^2 + \frac{b}{a}x + \frac{c}{a} \\ &= x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} \\ &= \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}. \end{aligned}$$

Multiplying both sides by a we get

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}. \quad (1)$$

Regardless of the sign of a, the RHS has it's extreme points when $x = -\frac{b}{2a}$ since this is when the square term disappears. Note that if $a \geq 0$ (i.e. convex parabola), the apex is at the lowest point of the parabola; otherwise (i.e. concave), the apex is at a peak.





Method 2. An alternative approach is to note that the apex occurs at the axis of symmetry of the parabola. Inspecting (1), taking points $x = -\frac{b}{2a} \pm d$ (for any distance $d > 0$) yields the same y values. Hence, $x = -\frac{b}{2a}$ *must* be the axis of symmetry of the parabola.

Method 3. The last approach relies on some calculus. If you haven't done this yet, just ignore this for now.

Let $f(x) = ax^2 + bx + c$. Taking the derivative, we get

$$f'(x) = 2ax + b$$

and the function achieves its extreme points when this is zero; i.e. $x = -\frac{b}{2a}$.

Regardless of the method we take to determine the x-coordinate of the apex, we can plug this into the equation for the parabola to get apex's y-coordinates. The full coordinate is

$$\Big(-\frac{b}{2a}, \frac{4ac - b^2}{4a}\Big).$$

Problem 3. *Solve, for any real numbers a, b and c, the general quadratic equation*

$$ax^2 + bx + c = 0.$$

Don't just use the quadratic formula, derive it! [Hint: complete the square].

Solution: We arrived at (1) by completing the square. Setting y to zero and arranging the equation we get

$$\frac{b^2 - 4ac}{4a^2} = \Big(x + \frac{b}{2a}\Big)^2$$

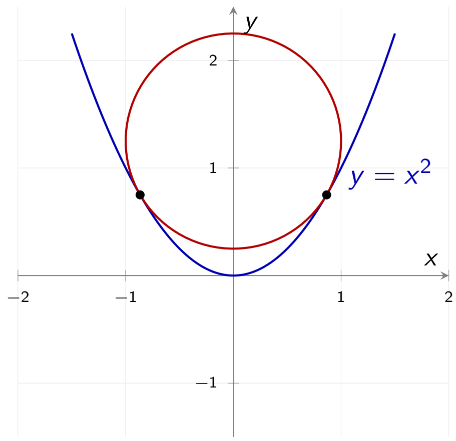
which happens if and only if

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

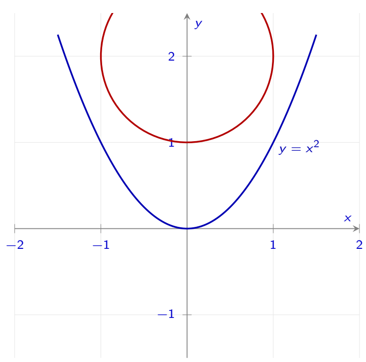
Note that, if the roots exist, their midpoints are $-\frac{b}{2a}$!

Problem 4. *Consider a circle of radius 1 centred at a point $(0, h)$ (h is an arbitrary real number) and a parabola consisting of all the points (x, y) satisfying $y = x^2$.*

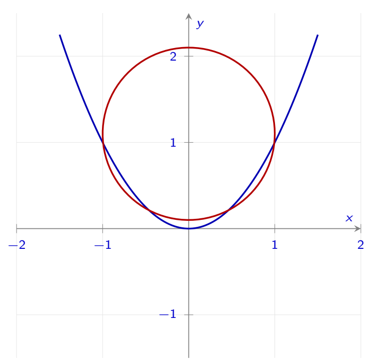
- *Try to visualise what happens as you vary h. Do the number of points of intersection between the circle and parabola change as you do so? Describe the important cases in words.*
- *Provide a general (algebraic) expression for the points of intersection between the circle and parabola for arbitrary h. Does it align with your expections? Should it? If so, why?*
- *For what value(s) of h are the circle and parabola tangent to one another? I.e. for what h does the situation portrayed below happen? Provide the coordinates of the points of intersection.*



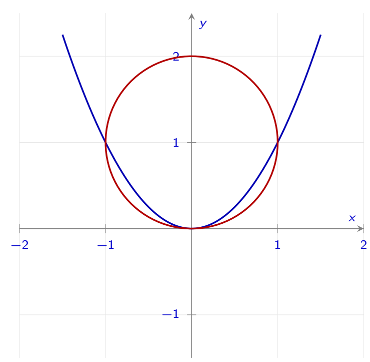
Solution: There are six cases other than the special case portrayed in the question.



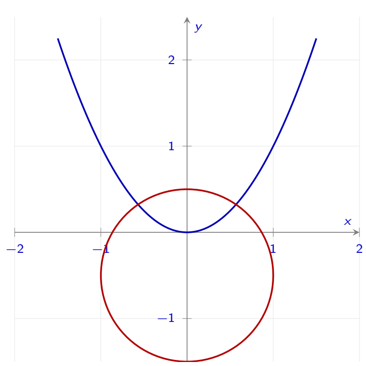
(a) $h > 5/4$ (no intercepts)



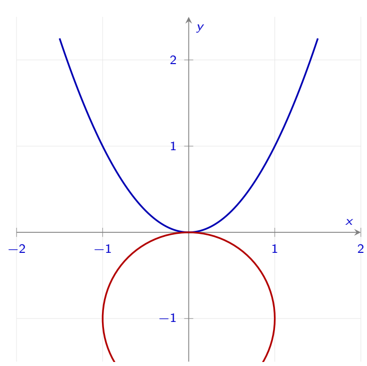
(b) $1 < h < 5/4$ (4 intercepts)



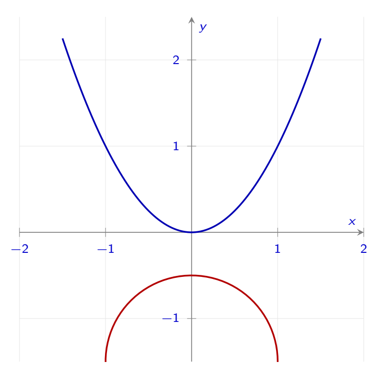
(c) $h = 1$ (3 intercepts)



(d) $-1 < h < 1$ (2 intercepts)



(e) $h = -1$ (1 intercept)



(f) $h < -1$ (no intercepts)

To get the general algebraic expression for the intercepts, we need to find all points (x, y) that simultaneously satisfy

$$x^2 + (y-h)^2 = 1 \quad \text{(Circle of radius 1 centered at } (0,h)\text{)} \tag{2}$$
$$y = x^2 \quad \text{(Standard upward-opening parabola).} \tag{3}$$

Caution! since $y = x^2$ is always non-negative, we need to be careful not to blindly substitute (3) into (2). I.e. it would be *incorrect* to simply compute the roots of the quadratic equation

$$y + (y-h)^2 = 1.$$

What we actually need is to solve the quadratic equation for *non-negative* roots

$$y + (y-h)^2 = 1, \quad y \geq 0.$$

Again, keeping a visual picture should make this obvious. Expanding, and using the quadratic equation, yields the general formula for arbitrary h:

$$y = \frac{(2h-1) \pm \sqrt{5-4h}}{2}, \quad y \geq 0. \tag{4}$$

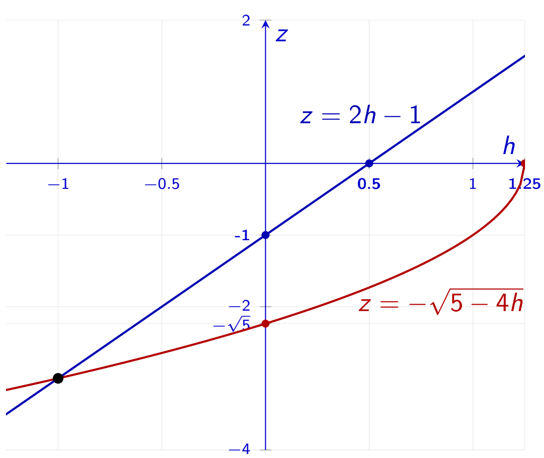
Note! While this looks like two roots, the number of roots varies with differing values of h. We'll study the cases in a moment.

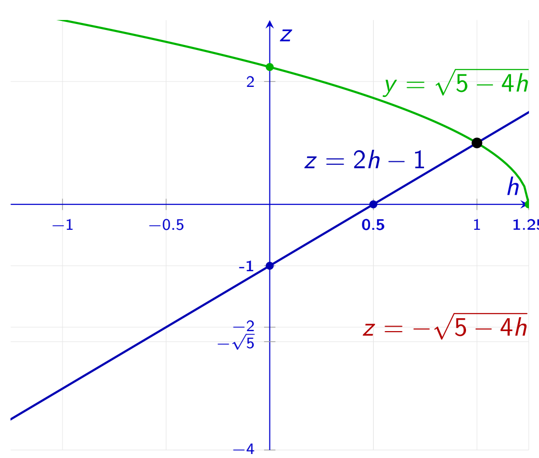
Once we've determined any solution y^* to (4), we can then immediately compute the x-coordinates of the points of interception as $x^* = \pm\sqrt{y^*}$.

Lets now check if the algebraic expressions line up with our expectations.

If $h > 5/4$ the discriminant $5 - 4h$ is negative, so there can be no real solutions (i.e. no intercepts). So we only need to consider the sub-cases of $h \leq 5/4$.

First, we'd like to know when the first quantity $y = \frac{(2h-1)+\sqrt{5-4h}}{2}$ in (4) is non-negative—i.e. $2h - 1 \geq -\sqrt{5-4h}$. It's clear from a basic plot that the main point of interest is when $2h-1 = -\sqrt{5-4h}$ for $h < 0$. It's necessary for h to satisfy $5 - 4h = (2h-1)^2$ which is satisfied if and only if $h^2 = 1$ whose only negative root is $h = -1$. Hence for $-1 \leq h \leq 5/4$, $y = \frac{(2h-1)+\sqrt{5-4h}}{2} \geq 0$.





What about the second quantity $y = \frac{(2h-1)-\sqrt{5-4h}}{2}$? Arguing similarly to before, we'll find that this expression is non-negative when $1 \leq h \leq 5/4$.

This matches our expectations as portrayed visually.

Some special cases:

- When $h = 1$, then $y = \frac{(2h-1)-\sqrt{5-4h}}{2} = 0$ which makes sense of the 3 intercepts portrayed in case (c) (why?).
- When $h = -1$, $y = \frac{(2h-1)+\sqrt{5-4h}}{2} = 0$, which explains the single intercept in (e).
- For $h < -1$, both roots in (4) are negative, which explains (f).

Finally, when $h = 5/4$, we get from (4) that $y = 3/4 \geq 0$ and so the tangent points of contact are

$$\left(\pm \frac{\sqrt{3}}{2}, \frac{3}{4} \right).$$